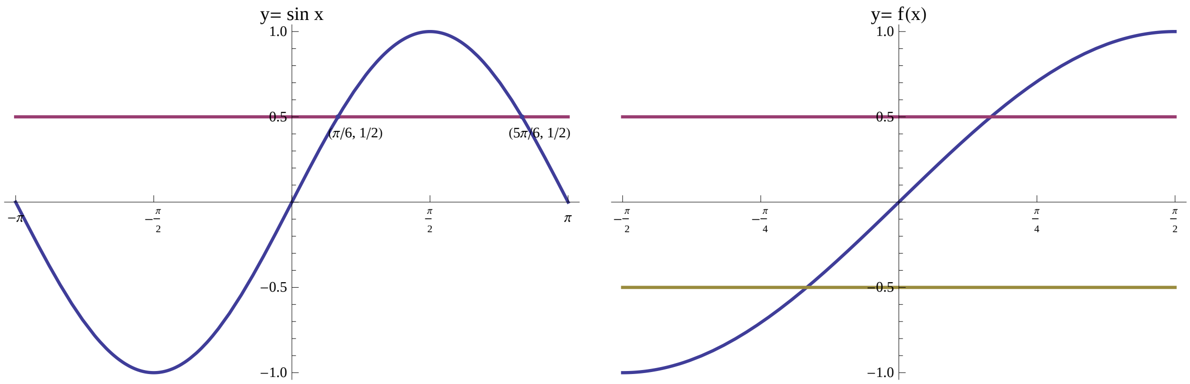
**Lecture 6 : Inverse Trigonometric Functions**

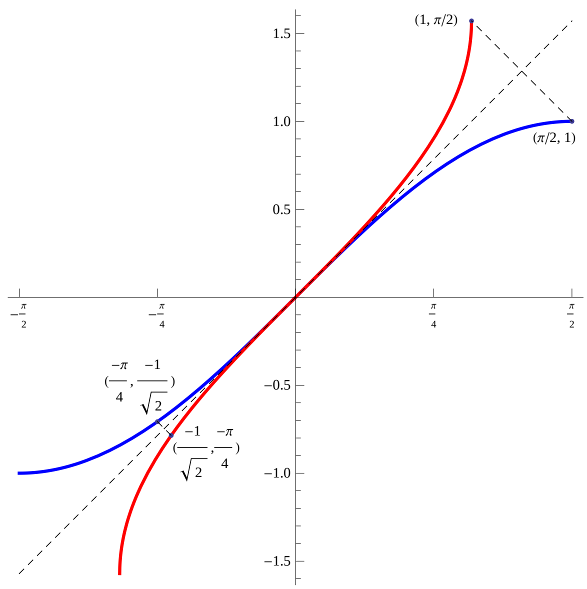
**Inverse Sine Function (arcsin x =** $sin^{-1}x$**)** The trigonometric function $\sin x$ is not one-to-one functions, hence in order to create an inverse, we must restrict its domain.
**The restricted sine function** is given by

$$f(x) = \begin{cases} \sin x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We have Domain(f) $= [-\frac{\pi}{2}, \frac{\pi}{2}]$ and Range(f) $= [-1, 1]$.




We see from the graph of the restricted sine function (or from its derivative) that the function is one-to-one and hence has an inverse, shown in red in the diagram below.




This inverse function, $f^{-1}(x)$, is denoted by

$$\boxed{f^{-1}(x) = \sin^{-1} x \quad \text{or} \quad \arcsin x.}$$

**Properties of** $\sin^{-1} x$**.**

Domain($\sin^{-1}$) $= [-1, 1]$ and Range($\sin^{-1}$) $= [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Since $f^{-1}(x) = y$ if and only if $f(y) = x$, we have:

$$\boxed{\sin^{-1} x = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.}$$

Since $f(f^{-1})(x) = x \quad f^{-1}(f(x)) = x$ we have:

$$\boxed{\sin(\sin^{-1}(x)) = x \text{ for } x \in [-1,1] \quad \sin^{-1}(\sin(x)) = x \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].}$$

from the graph: $\sin^{-1} x$ is an odd function and $\sin^{-1}(-x) = -\sin^{-1} x$.

**Example** Evaluate $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ using the graph above.

**Example** Evaluate $\sin^{-1}(\sqrt{3}/2)$, $\sin^{-1}(-\sqrt{3}/2)$,

**Example** Evaluate $\sin^{-1}(\sin \pi)$.

**Example** Evaluate $\cos(\sin^{-1}(\sqrt{3}/2))$.

**Example** Give a formula in terms of $x$ for $\tan(\sin^{-1}(x))$

## Derivative of $\sin^{-1} x$.

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad -1 \le x \le 1.}$$

**Proof** We have $\sin^{-1} x = y$ if and only if $\sin y = x$. Using implicit differentiation, we get $\cos y \frac{dy}{dx} = 1$ or

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

Now we know that $\cos^2 y + \sin^2 y = 1$, hence we have that $\cos^2 y + x^2 = 1$ and

$$\cos y = \sqrt{1-x^2}$$

and

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}.$$

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If we use the chain rule in conjunction with the above derivative, we get

$$\boxed{\frac{d}{dx}\sin^{-1}(k(x)) = \frac{k'(x)}{\sqrt{1-(k(x))^2}}, \quad x \in \text{Dom}(k) \text{ and } -1 \le k(x) \le 1.}$$

**Example** Find the derivative

$$\frac{d}{dx}\sin^{-1}\sqrt{\cos x}$$

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**Inverse Cosine Function** We can define the function $\cos^{-1}x = \arccos(x)$ similarly. The details are given at the end of this lecture.

$$\text{Domain}(\cos^{-1}) = [-1,1] \quad \text{and} \quad \text{Range}(\cos^{-1}) = [0,\pi].$$

$$\boxed{\cos^{-1}x = y \quad \text{if and only if} \quad \cos(y) = x \text{ and } 0 \le y \le \pi.}$$

$$\boxed{\cos(\cos^{-1}(x)) = x \text{ for } x \in [-1,1] \quad \cos^{-1}(\cos(x)) = x \text{ for } x \in \big[0,\pi\big].}$$

It is shown at the end of the lecture that

$$\frac{d}{dx}\cos^{-1}x = -\frac{d}{dx}\sin^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

and one can use this to prove that

$$\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}.$$

## Inverse Tangent Function

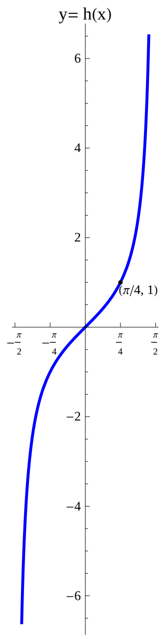
The tangent function is not a one to one function, however we can also restrict the domain to construct a one to one function in this case.

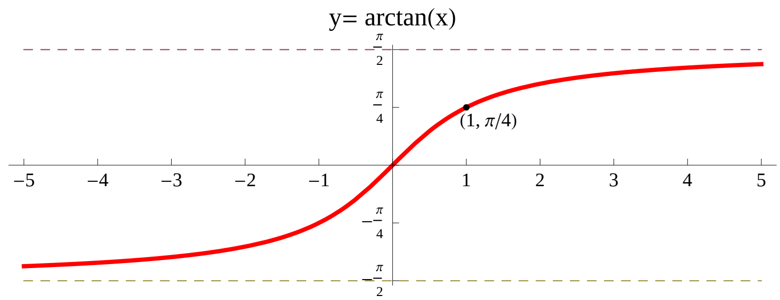
**The restricted tangent function** is given by

$$h(x) = \begin{cases} \tan x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We see from the graph of the restricted tangent function (or from its derivative) that the function is one-to-one and hence has an inverse, which we denote by

$$\boxed{h^{-1}(x) = \tan^{-1}x \quad \text{or} \quad \arctan x.}$$




**Properties of $tan^{-1}x$.**

Domain$(\tan^{-1}) = (-\infty, \infty)$ and Range$(\tan^{-1}) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

Since $h^{-1}(x) = y$ if and only if $h(y) = x$, we have:

$$\boxed{\tan^{-1} x = y \text{ if and only if } \tan(y) = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.}$$

Since $h(h^{-1}(x)) = x$ and $h^{-1}(h(x)) = x$, we have:

$$\boxed{\tan(\tan^{-1}(x)) = x \text{ for } x \in (-\infty, \infty) \quad \tan^{-1}(\tan(x)) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).}$$

Frpm the graph, we have: $\boxed{\tan^{-1}(-x) = -\tan^{-1}(x).}$

Also, since $\lim_{x \to (\frac{\pi}{2}^-)} \tan x = \infty$ and $\lim_{x \to (-\frac{\pi}{2}^+)} \tan x = -\infty$,

we have $\boxed{\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}}$ and $\boxed{\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}}$

**Example** Find $\tan^{-1}(1)$ and $\tan^{-1}(\frac{1}{\sqrt{3}})$.

**Example** Find $\cos(\tan^{-1}(\frac{1}{\sqrt{3}}))$.

**Derivative of $\tan^{-1} x$.**

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}, \quad -\infty < x < \infty.}$$

**Proof** We have $\tan^{-1} x = y$ if and only if $\tan y = x$. Using implicit differentiation, we get $\sec^2 y \frac{dy}{dx} = 1$ or

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y.$$

Now we know that $\cos^2 y = \cos^2(\tan^{-1} x) = \frac{1}{1+x^2}$. proving the result.

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If we use the chain rule in conjunction with the above derivative, we get

$$\boxed{\frac{d}{dx} \tan^{-1}(k(x)) = \frac{k'(x)}{1 + (k(x))^2}, \quad x \in \text{Dom}(k)}$$

**Example** Find the domain and derivative of $\tan^{-1}(\ln x)$

Domain $= (0, \infty)$

$$\frac{d}{dx} \tan^{-1}(\ln x) = \frac{\frac{1}{x}}{1 + (\ln x)^2} = \frac{1}{x(1 + (\ln x)^2)}$$

**Integration formulas**

Reversing the derivative formulas above, we get

$$\boxed{\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C, \qquad \int \frac{1}{x^2+1}\, dx = \tan^{-1} x + C,}$$

**Example**

$$\int \frac{1}{\sqrt{9-x^2}}\, dx =$$

$$\int \frac{1}{3\sqrt{1-\frac{x^2}{9}}}\, dx = \int \frac{1}{3\sqrt{1-\frac{x^2}{9}}} dx = \frac{1}{3} \int \frac{1}{\sqrt{1-\frac{x^2}{9}}} dx$$

Let $u = \frac{x}{3}$, then $dx = 3du$ and

$$\int \frac{1}{\sqrt{9-x^2}}\, dx = \frac{1}{3} \int \frac{3}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1} \frac{x}{3} + C$$

**Example**

$$\int_0^{1/2} \frac{1}{1+4x^2}\, dx$$

Let $u = 2x$, then $du = 2dx$, $\quad u(0) = 0$, $\quad u(1/2) = 1$ and

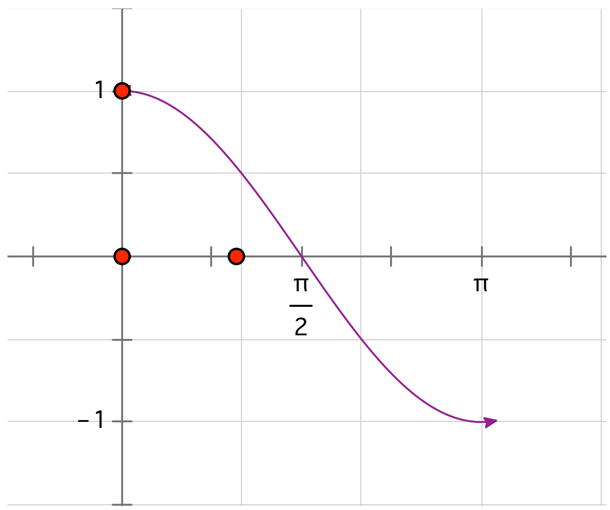
$$\int_0^{1/2} \frac{1}{1+4x^2}\, dx = \frac{1}{2} \int_0^1 \frac{1}{1+u^2}\, dx = \frac{1}{2} \tan^{-1} u|_0^1 = \frac{1}{2}[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$\frac{1}{2}[\frac{\pi}{4} - 0] = \frac{\pi}{8}.$$

**The restricted cosine function** is given by

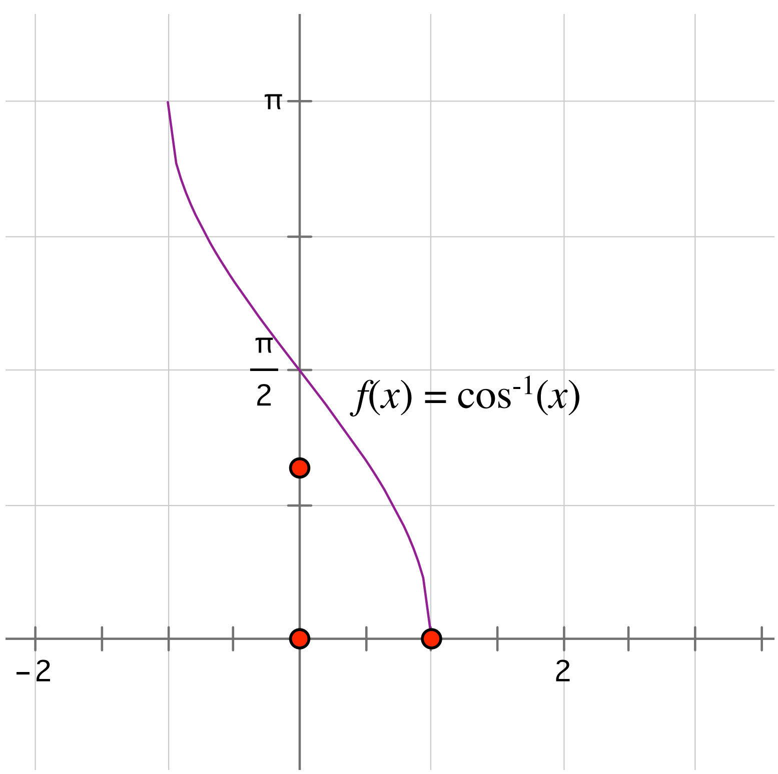
$$g(x) = \begin{cases} \cos x & 0 \le x \le \pi \\ \text{undefined} & \text{otherwise} \end{cases}$$

We have Domain(g) $= [0, \pi]$ and Range(g) $= [-1, 1]$.




We see from the graph of the restricted cosine function (or from its derivative) that the function is one-to-one and hence has an inverse,

$$\boxed{g^{-1}(x) = \cos^{-1} x \quad \text{or} \quad \arccos x}$$



$$\text{Domain}(\cos^{-1}) = [-1, 1] \quad \text{and} \quad \text{Range}(\cos^{-1}) = [0, \pi].$$

Recall from the definition of inverse functions:

$$g^{-1}(x) = y \text{ if and only if } g(y) = x.$$

$$\boxed{\cos^{-1} x = y \quad \text{if and only if} \quad \cos(y) = x \text{ and } 0 \le y \le \pi.}$$

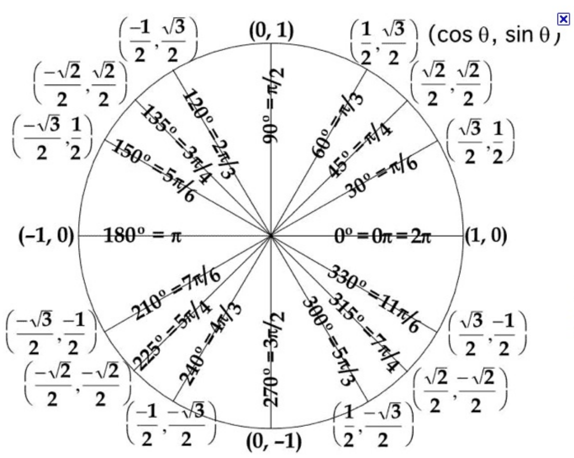
$$g(g^{-1}(x)) = x \quad g^{-1}(g(x)) = x$$

$$\boxed{\cos(\cos^{-1}(x)) = x \text{ for } x \in [-1, 1] \quad \cos^{-1}(\cos(x)) = x \text{ for } x \in [0, \pi].}$$

Note from the graph that $\boxed{\cos^{-1}(-x) = \pi - \cos^{-1}(x)}$.

$\cos^{-1}(\sqrt{3}/2) =$ ______ and $\cos^{-1}(-\sqrt{3}/2) =$ ______

You can use either chart below to find the correct angle between 0 and $\pi$.:

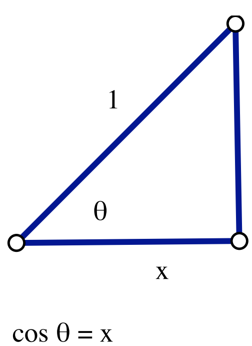


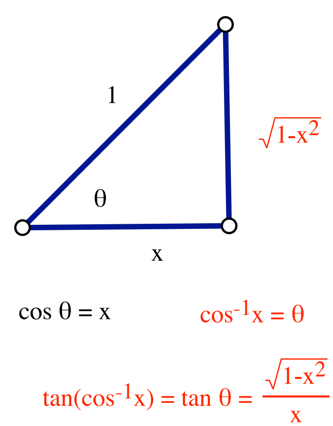

| | 0° | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|
| sin | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | −1 | 0 |
| cos | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | −1 | 0 | 1 |
| tan | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | not defined | 0 | not defined | 0 |

$\tan(\cos^{-1}(\sqrt{3}/2)) =$ ______

$\tan(\cos^{-1}(x)) =$ ______

Must draw a triangle with correct proportions:



$$\boxed{\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}, \quad -1 \le x \le 1.}$$

**Proof** We have $\cos^{-1}x = y$ if and only if $\cos y = x$. Using implicit differentiation, we get $-\sin y\frac{dy}{dx} = 1$ or

$$\frac{dy}{dx} = \frac{-1}{\sin y}.$$

Now we know that $\cos^2 y + \sin^2 y = 1$, hence we have that $\sin^2 y + x^2 = 1$ and

$$\sin y = \sqrt{1-x^2}$$

and

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}.$$

**Note** that $\frac{d}{dx}\cos^{-1}x = -\frac{d}{dx}\sin^{-1}x$. In fact we can use this to prove that $\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$.

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If we use the chain rule in conjunction with the above derivative, we get

$$\boxed{\frac{d}{dx}\cos^{-1}(k(x)) = \frac{-k'(x)}{\sqrt{1-(k(x))^2}}, \quad x \in \mathrm{Dom}(k) \text{ and } -1 \le k(x) \le 1.}$$